(see continuation of problem from Monday in Lecture 2 notes)

§1.5 Linear 1ST Order Equations

recall separable eq'n: $\frac{dy}{dx} = 2xy \qquad y > 0$

$\downarrow$ sep.

$\boxed{\frac{1}{y}}\, dy = \boxed{2x}\, dx$ $\qquad \square$ recognize both as derivatives

$D_x(\ln y) = D_x(x^2)$

instead of division, use multiplication to separate,

i.e., multiply both sides by integrating factor

s.t. both sides can be recognized as derivatives

integrating factor for DE is function: $\rho(x,y)$ ← rho

in previous example, integrating factor $\rho(y) = \frac{1}{y}$ ← only dependent on y

explore how $\rho(x,y)$ used to solve linear 1ST order eq'ns

format: $\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$ where $P(x), Q(x)$ are coefficient functions

continuous on interval I

revisit $\frac{dy}{dx} = 2xy \Rightarrow \frac{dy}{dx} \overbrace{- 2x}^{P(x)} y = \overset{Q(x)}{0}$ $\quad \therefore$ linear DE

integrating factor for general linear eq'n: $\rho(x) = e^{\int P(x)dx}$

assuming necessary antiderivatives can be found

$$\underbrace{\frac{dy}{dx}}_{f'}\underbrace{e^{\int P(x)dx}}_{g} + \underbrace{P(x)e^{\int P(x)dx}}_{g'}\underbrace{y}_{f} = Q(x)\,e^{\int P(x)dx}$$

$$D_x(fg)$$

$$D_x\left[y \cdot e^{\int P(x)dx}\right] = Q(x)\,e^{\int P(x)dx}$$

side notes:

$D_x\left[\int P(x)dx\right] = P(x)$

$\left(e^{\int P(x)dx}\right)' \qquad u = \int P(x)dx,\ u' = P(x)$

$= e^u u'$

$= e^{\int P(x)dx} \cdot P(x)$

Product Rule: $(fg)' = f'g + fg'$

integration results in: $\quad y \cdot e^{\int P(x)dx} = \int \left(Q(x)e^{\int P(x)dx}\right)dx + C$

solve for y by $\quad y(x) = e^{-\int P(x)dx}\left[\int \left(Q(x)e^{\int P(x)dx}\right)dx + C\right]$

$(fg)' = f'g + fg'$

solve for y by mult by $e^{-\int P(x)dx}$ $\quad y(x) = e^{-\int P(x)dx}\left[\int \left(Q(x)e^{\int P(x)dx}\right)dx + C\right]$

don't memorize - just understand the mechanics

ex. solve $\underbrace{\frac{dy}{dx} - y}_{P(x)=-1} = \underbrace{\frac{11}{8}e^{-x/3}}_{Q(x)}$ given $y(0) = -1$

$$\rho(x) = e^{\int P(x)dx} = e^{\int(-1)dx} = e^{-x} \longleftarrow \text{integrating factor}$$

$$\underbrace{e^{-x}}_{g}\underbrace{\frac{dy}{dx}}_{f'} - \underbrace{e^{-x}}_{g'}\underbrace{y}_{f} = \frac{11}{8}\underbrace{e^{-x/3}e^{-x}}$$

$$\frac{d}{dx}[fg]$$

$$\frac{d}{dx}\left(y\cdot e^{-x}\right) = \frac{11}{8}e^{-4x/3}$$

$$\int e^{-4x/3}dx \quad u = -\frac{4x}{3} \quad -\frac{3}{4}du = dx$$

$$y\cdot e^{-x} = \frac{11}{8}\int e^{-4x/3}dx = \frac{11}{8}\left(-\frac{3}{4}\right)\int e^u du$$

$$e^x\left(y\cdot e^{-x} = -\frac{33}{32}e^{-4x/3} + C\right)$$

$$y = -\frac{33}{32}\underbrace{e^{-4x/3}e^x} + Ce^x$$

$$\boxed{y(x) = -\frac{33}{32}e^{-x/3} + Ce^x} \text{ general solution}$$

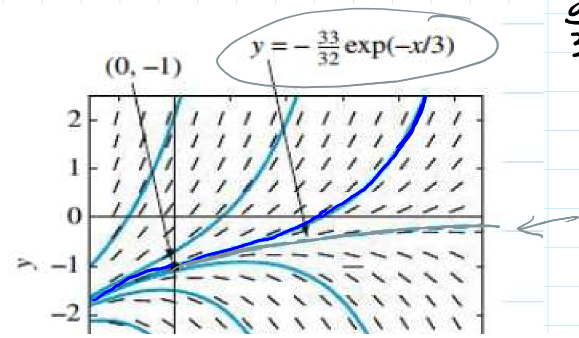
$y(0) = -1$: $\quad y(0) = -1 = -\frac{33}{32}\cdot 1 + C$

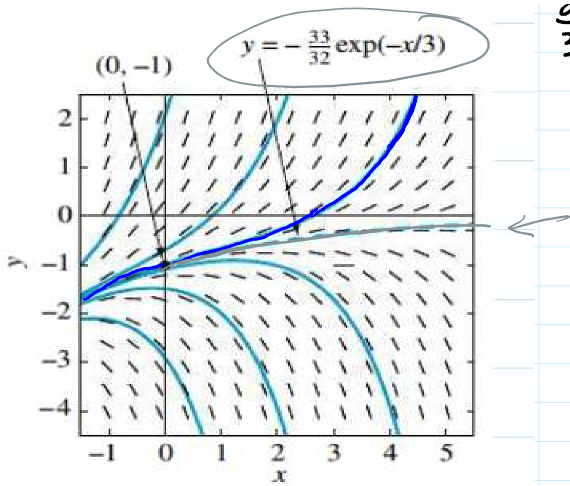
$$-\frac{32}{32} + \frac{33}{32} = C$$

$$\frac{1}{32} = C \Rightarrow \boxed{y(x) = -\frac{33}{32}e^{-x/3} + \frac{1}{32}e^x}$$

OR

$$\boxed{y(x) = \frac{1}{32}\left(e^x - 33e^{-x/3}\right)}$$





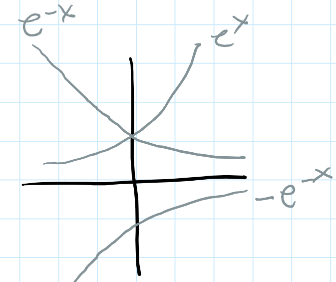
$\frac{\ }{32} + \frac{\ }{32} = C$

$\frac{1}{32} = C \Rightarrow \boxed{y(x) = -\frac{33}{32}e^{-x/3} + \frac{1}{32}e^{x}}$

OR

$\boxed{y(x) = \frac{1}{32}\left(e^{x} - 33e^{-x/3}\right)}$

when $y(0) = -1$ ← y_0



when $C = 0$: $y = -\frac{33}{32}e^{-x/3}$

initial condition

• when $y_0 > -\frac{33}{32}$ $C > 0$

$\Downarrow$

e^x will dominate RHS

as $x \to +\infty$, $y \to +\infty$

• when $y_0 < -\frac{33}{32}$ $C < 0$ as $x \to +\infty$, $y \to -\infty$

we consider $y_0 = -\frac{33}{32}$ a critical condition

ex. find general solution of $(x^2+1)\frac{dy}{dx} + 3xy = 6x$

can divide by x^2+1
bc it can never be zero
(will never cause UND)

$\frac{dy}{dx} + \boxed{\frac{3x}{x^2+1}}y = \boxed{\frac{6x}{x^2+1}}$

$P(x)$ $Q(x)$

$\downarrow$

$\rho(x) = e^{\int P(x)dx}$

$= e^{3\int \frac{x}{x^2+1}dx}$

$= e^{\frac{3}{2}\ln(x^2+1)}$

$= e^{\ln(x^2+1)^{3/2}}$

$\rho(x) = (x^2+1)^{3/2}$

$u = x^2+1$

$\frac{1}{2}du = dx$

$\frac{1}{2}\int \frac{1}{u}du$

$\frac{1}{2}\ln|u|$

$\frac{1}{2}\ln(x^2+1)$

always > 0

$(x^2+1)^{3/2}\frac{dy}{dx} + \frac{3x}{x^2+1}\cdot(x^2+1)^{3/2}\cdot y = (x^2+1)^{3/2}\cdot\frac{6x}{x^2+1}$

$$(x^2+1)^{3/2}\frac{dy}{dx} + \frac{3x}{x^2+1}\cdot(x^2+1)^{3/2}\cdot y = (x^2+1)^{3/2}\cdot\frac{6x}{x^2+1}$$

always > 0

$$(x^2+1)^{3/2}\frac{dy}{dx} + 3x\,(x^2+1)^{1/2}\,y = (x^2+1)^{1/2}\cdot 6x$$

$$\frac{d}{dx}\left(y\cdot(x^2+1)^{3/2}\right) = 6x\,(x^2+1)^{1/2}$$

$$y\,(x^2+1)^{3/2} = 6\int (x^2+1)^{1/2}\,x\,dx$$

$u = x^2+1$

$$= 3\int u^{1/2}\,du$$

$$= \cancel{3}\cdot\frac{2}{\cancel{3}}\,u^{3/2} + C$$

$$= 2\,(x^2+1)^{3/2} + C$$

$$= 2 + C\,(x^2+1)^{-3/2}$$

$$\boxed{y(x) = 2 + C\,(x^2+1)^{-3/2}}$$